مثال

فرض کنید z تابعی از x, y باشد که توسط معادله‌ی ضمنی زیر داده شده است

$$\cos(xyz) = x^2 + y^2 + z^2 - 1$$

مطلوب است محاسبه‌ی $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

یادآوری

$$F(x, y, z) = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

$$\cos(xyz) - x^2 - y^2 - z^2 + 1 = 0$$

$$\frac{\partial z}{\partial x} = -\frac{-yz\sin(xyz) - 2x}{-xy\sin(xyz) - 2z}$$

$$\frac{\partial z}{\partial y} = -\frac{-xz\sin(xyz) - 2y}{-xy\sin(xyz) - 2z}$$

قضیه‌ی تابع ضمنی

فرض کنید $F(x,y,z)$ در یک گوی شامل (a,b,c) تعریف شده باشد و $F(a,b,c)=0$ و $F_z(a,b,c) \neq 0$، آنگاه از معادله‌ی $F(x,y,z)=0$ می‌توان z را به صورت تابعی از x, y استخراج کرد و برای این تابع داریم

$$\frac{\partial z}{\partial x}(a,b) = \frac{-F_x}{F_z}$$

$$\frac{\partial z}{\partial y}(a,b) = \frac{-F_y}{F_z}$$

مثال

فرض کنید

$$\mathbb{R}^2 \xrightarrow{\frac{xy}{x^2+y^2}} \mathbb{R} \xrightarrow{f} \mathbb{R}$$

$$z = f\left(\frac{xy}{x^2+y^2}\right)$$

نشان دهید که

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial x} = \frac{y(x^2+y^2) - 2x(xy)}{(x^2+y^2)^2} \times f'\left(\frac{xy}{x^2+y^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{x(x^2+y^2) - 2y(xy)}{(x^2+y^2)^2} \times f'\left(\frac{xy}{x^2+y^2}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{xy(x^2+y^2) - 2x^2(xy)}{(x^2+y^2)^2} \times f'\left(\frac{xy}{x^2+y^2}\right) + \frac{xy(x^2+y^2) - 2y^2(xy)}{(x^2+y^2)^2} \times f'\left(\frac{xy}{x^2+y^2}\right) =$$

$$f'\left(\frac{xy}{x^2+y^2}\right)\left(\frac{2xy(x^2+y^2) - 2xy(x^2+y^2)}{(x^2+y^2)^2}\right) = 0$$

$\mathbb{R}^2 \xrightarrow{g(x,y)} \mathbb{R} \xrightarrow{f} \mathbb{R}$

$z(x,y) = f \circ g(x,y)$

$\frac{\partial z}{\partial x} = \frac{\partial g}{\partial x} \times f'(g(x,y))$

$\frac{\partial z}{\partial y} = \frac{\partial g}{\partial y} \times f'(g(x,y))$

مثال

فرض کنید $f:\mathbb{R}^2 \to \mathbb{R}$ تابعی مشتق‌پذیر (دیفرانسیل‌پذیر) باشد، و z تابعی از x, y باشد که در معادلهٔ ضمنی زیر صدق می‌کند:

$$f(\underbrace{x\sqrt{z}}_{u}, \underbrace{y\sqrt{z}}_{v}) = 0$$

نشان دهید که

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -2z$$

توجه

اگر می‌دانستیم

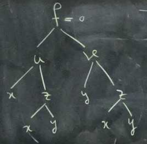
$$F(x,y,z) = 0$$

آنگاه

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

حل سوال بدون استفاده از فرمول

$$\frac{\partial f}{\partial x} = 0$$



$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} =$$

$$\frac{-x\frac{\partial f}{\partial u}\sqrt{z} - y\frac{\partial f}{\partial v}\sqrt{z}}{\frac{\partial f}{\partial u}\frac{x}{2\sqrt{z}} + \frac{\partial f}{\partial v}\frac{y}{2\sqrt{z}}} = -2z$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z}\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}\frac{x}{2\sqrt{z}}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial v}\sqrt{z} + \frac{\partial f}{\partial v}\frac{y}{2\sqrt{z}}\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y}\left(\frac{\partial f}{\partial u}\frac{x}{2\sqrt{z}} + \frac{\partial f}{\partial v}\frac{y}{2\sqrt{z}}\right) = -\frac{\partial f}{\partial v}\sqrt{z}$$

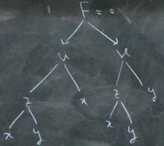
$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-\frac{\partial f}{\partial v}\sqrt{z}}{\frac{\partial f}{\partial u}\frac{x}{2\sqrt{z}} + \frac{\partial f}{\partial v}\frac{y}{2\sqrt{z}}}$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}\sqrt{z} + \frac{\partial f}{\partial u}\frac{x}{2\sqrt{z}}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial v}\frac{y}{2\sqrt{z}}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(\frac{\partial f}{\partial u}\frac{x}{2\sqrt{z}} + \frac{\partial f}{\partial v}\frac{y}{2\sqrt{z}}\right) = -\frac{\partial f}{\partial u}\sqrt{z} \Rightarrow \frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial u}\sqrt{z}}{\frac{\partial f}{\partial u}\frac{x}{2\sqrt{z}} + \frac{\partial f}{\partial v}\frac{y}{2\sqrt{z}}}$$

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -2z}$$



مثال

فرض کنید z تابعی از x, y باشد و

$$F(\underbrace{z-3x}_{u}, \underbrace{z-2y}_{v}) = 1$$

نشان دهید که

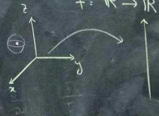
$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 6$$

پیوستگی

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$$

$$\forall \varepsilon>0 \ \exists \delta>0 \ \forall (x,y,z) \left(\sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta \rightarrow |f(x,y,z)-f(a,b,c)| < \varepsilon \right)$$

$f: \mathbb{R}^3 \to \mathbb{R}$



تمرین -

فرض کنید $f(x+z, y+z) = 1$

نشان دهید $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = -1$

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = \frac{6\left(\frac{\partial F}{\partial v} + \frac{\partial F}{\partial u}\right)}{\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}} = 6$$

$$\frac{\partial F}{\partial u} \times 1 \times \frac{\partial z}{\partial y} + \frac{\partial F}{\partial v} \cdot \frac{\partial z}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial v}(-2)}{\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}}$$

$$\frac{\partial F}{\partial u} \times \frac{\partial z}{\partial x} + \frac{\partial F}{\partial u}(-3) + \frac{\partial F}{\partial v} \times \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-\left(\frac{\partial F}{\partial u}(-3)\right)}{\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}}$$

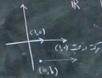
یادآوری

$$z = f(x,y)$$

$$(a,b) \in D(f)$$

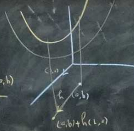
$$\frac{\partial z}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\frac{\partial z}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$



$$(a,b) + h(1,0)$$

$$\frac{\partial z}{\partial x}(a,b) = D_{\vec{u}=(1,0)} f(a,b) = \lim_{h \to 0} \frac{f\big((a,b)+h(1,0)\big) - f(a,b)}{h}$$



$$\frac{\partial z}{\partial y}(a,b) = D_{\vec{u}=(0,1)} f(a,b) = \lim_{h \to 0} \frac{f\big((a,b)+h(0,1)\big) - f(a,b)}{h}$$

مشتقات سوئی

$z = f(x,y)$

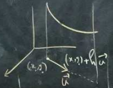
فرض کنید یک بردار یکه‌ی دلخواه باشد.

$\vec{u} = (c,d)$

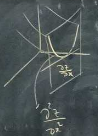
مشتق تابع z در نقطه (x_0,y_0) در جهت بردار $\vec{u}$

به صورت زیر نشان داده و تعریف می‌شود.

$$D_{\vec{u}} f(x_0,y_0) =$$



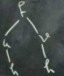
$$D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f\left((x_0,y_0)+h\vec{u}\right) - f(x_0,y_0)}{h} = \lim_{h\to 0} \frac{f(x_0+hc,\, y_0+hd) - f(x_0,y_0)}{h}$$



$$g(h) = f(x+ah,\ y+bh)$$

$$g'(h)^{(\circ)} = \frac{\partial f}{\partial u}^{(x,y)} a + \frac{\partial f}{\partial v}^{(x,y)} b$$

$$\Rightarrow g'(h)(\circ) = \frac{\partial z}{\partial x}(x,y)\, a + \frac{\partial z}{\partial y}(x,y)\, b$$



$$\lim_{h \to 0} \frac{f(x+ah,\ y+bh) - f(x,y)}{h} = ?$$

$$g(h) := f(x+ah,\ y+bh)$$

$$g'(0) = ? \qquad g'(0) = \lim_{h \to 0} \frac{f(x+ah,\ y+bh) - f(x,y)}{h}$$

$$D_{\vec{u}} f(x,y) =$$

$$\frac{\partial z}{\partial x}(x,y)\, a + \frac{\partial z}{\partial y}(x,y)\, b$$

$$= \left[\frac{\partial z}{\partial x}(x,y),\ \frac{\partial z}{\partial y}(x,y)\right] \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{=\vec{u}} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (a,b)$$

قضیه

فرض کنید $z = f(x,y)$ تابعی دیفرانسیل‌پذیر از x و y باشد و $\vec{u} = (a,b)$ یک بردار یکه (دلخواه) باشد و $(x,y) \in D(f)$، آنگاه