حالت ابهام: 1^{∞}

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = \lim_{x\to a} e^{g(x)(f(x)-1)}$$

$$g(x)\ln f(x) = \frac{g(x)\overbrace{\ln f(x)}^{1}}{(f(x)-1)}(f(x)-1)$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to+\infty} e^{x\left(\frac{1}{x}\right)} = e$$

حل

یادآوری فرض کنید

$$\lim_{x\to a} f(x) = 1$$

$$\lim_{x\to a} g(x) = \infty$$

تمرین

$$\int x^{\frac{x}{2}}\,dx$$

$$\int 3^{\sin\theta}\cos\theta\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$\int 3^{u}\,du = \frac{3^{u}}{\ln 3} + C = \frac{3^{\sin\theta}}{\ln 3} + C$$

حل

تمرین

$$\log_b x = \frac{\ln x}{\ln b}$$

$$\ln x = \log_e x$$

$$\log_b x = y \iff b^{y} = x$$

یادآوری

تعریف ← $b^{x} = e^{x\ln b}$, $b>0$

$\log_b x \rightarrow$ معکوس تابع بالا

به نام خدا

اهداف

محاسبه‌ی انتگرال‌های

$$\int \frac{1}{\sqrt{1-x^2}}\,dx \qquad \int \frac{1}{1+x^2}\,dx$$

$$\int \frac{1}{\sqrt{1+x^2}}\,dx$$

درس

تابع

$\sin x$ را در نظر بگیرید

$$\sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$$

تابع وارون $\sin$ را با $\sin^{-1}$ یا $\arcsin$ نشان می‌دهیم

$$\sin^{-1}(x) : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$y = \sin^{-1} x \iff \sin y = x$$

$-1 \le x \le 1$

یادآوری

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

$f(x) = y$

$$(\sin^{-1})'(\sin x) = \frac{1}{\sin' x} = \frac{1}{\cos x}$$

$$= \frac{1}{\sqrt{1-\sin^2 x}}$$

$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos x = \sqrt{1-\sin^2 x}$

$$(\sin^{-1})'(y) = \frac{1}{\sqrt{1-y^2}}$$

$$(\sin^{-1})'(x) = \frac{1}{\sqrt{1-x^2}}$$

$\sin^{-1}x : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\lim_{x\to\sqrt{2}} f(x) =$

$\lim_{x\to\sqrt{2}} \sin^{-1}(x^2-1) = \frac{\pi}{2}$

$\lim_{x\to\sqrt{2}^+} \sin^{-1}(x^2-1) = \frac{\pi}{2}$

$f'(x) = 2x \times \frac{1}{\sqrt{1-(x^2-1)^2}}$

$(\sin^{-1})'(u) = u' \times \frac{1}{\sqrt{1-u^2}}$

$f(x) = \sin^{-1}(x^2-1)$

$-1 \leq x^2-1 \leq 1 \Rightarrow$

$0 \leq x^2 \leq 2 \Leftrightarrow$

$-\sqrt{2} \leq x \leq \sqrt{2}$

$\int \frac{1}{\sqrt{1-x^2}} dx =$

$\sin^{-1}x = \arcsin x$

$\sin^{-1}x : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}} > 0$

$\sin^{-1}(0) = 0$

$$(\sin^{-1})'(x) = \quad \text{توجه}$$

$$-(\cos^{-1})'x$$

توجه

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

اثبات:

$$\left(\sin^{-1}x + \cos^{-1}x\right)' = 0$$

پس $\sin^{-1}x + \cos^{-1}x$ یک تابع ثابت است

$$\sin^{-1}0 + \cos^{-1}0 = \frac{\pi}{2}$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x$$

$$\int \frac{1}{-\sqrt{1-x^2}}\,dx = \cos^{-1}x$$

$$= -\sin^{-1}x$$

$$(\cos^{-1})'(y) = \frac{1}{-\sqrt{1-y^2}}$$

سوال: آیا در بازه $[0,\pi]$

$$\sin x = \sqrt{1-\cos^2 x}$$

بله. زیرا در این بازه sin یک تابع مثبت است

$$\cos^{-1}(\cos x) = x$$

$$\cos(\cos^{-1}x) = x$$

$$(\cos^{-1})'(\cos x) = \frac{1}{\cos' x} = \frac{1}{-\sin x}$$

cos x را در بازه $[0,\pi]$ در نظر بگیرید

$$\cos : [0,\pi] \to [-1,1]$$

$$\cos^{-1} : [-1,1] \to [0,\pi]$$

تابع

مثال

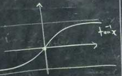
$$f(x) = \sin^{-1}(x^2-1)$$

دامنه f':

$$(-\sqrt{2},0)\cup(0,\sqrt{2})$$

$$f'(x) = \frac{2x}{\sqrt{1-(x^2-1)^2}}$$

$$\lim_{x\to-\infty} \tan^{-1}x = -\frac{\pi}{2}$$



$$\lim_{x\to+\infty} \tan^{-1}x = \frac{\pi}{2}$$

$$\left(x\to\frac{\pi}{2} \Rightarrow \tan x \to +\infty\right)$$

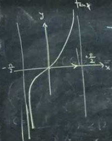
$$\left(y\to+\infty \Rightarrow \tan^{-1}y \to \frac{\pi}{2}\right)$$

$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, +\infty)$ تابع اکیداً صعودی است. لذا تعریف می‌کنیم:

$$\tan^{-1}x : (-\infty, +\infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\tan^{-1}x = y \Leftrightarrow \tan y = x \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

$$\tan^{-1}0 = 0$$



مثال

$$\lim_{x\to+\infty} \arccos\left(\frac{1+x^2}{1+2x^2}\right)$$

$$= \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

مثال

$$\cos^{-1}x : [-1,1] \to [0,\pi]$$

$$\left(\cos^{-1}x\right)' = \frac{-1}{\sqrt{1-x^2}}$$

$\int \frac{1}{\sqrt{1-ax^2}}\,dx$

$u=\sqrt{a}\,x$

$\frac{1}{\sqrt{a}}\sin^{-1}(\sqrt{a}\,x)+c$

$\int_0^{\frac14} \frac{1}{\sqrt{1-4x^2}}\,dx$ dis

$2x=u$

$du=2\,dx$

$u=\frac{1}{2}$

$\int_0^{\frac12} \frac{du}{2\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}(u)\Big|_0^{\frac12}$

$=\frac{1}{2}\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

$\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)=\frac{\pi}{2}$ dis

$(\sin^{-1})'(x)=\frac{1}{\sqrt{1-x^2}}$

$(\tan^{-1})'(x)=\frac{1}{1+x^2}$

$1+\tan^2 y=\frac{1}{\cos^2 y}$

$1+x^2=\frac{1}{\cos^2 y}$

$\cos y=\frac{1}{\sqrt{1+x^2}}$

$\cos(\tan^{-1}x)=?$ dis

$y=\tan^{-1}x \Rightarrow \cos(y)=?$

$\tan y = x$

$\frac{a}{b}=x$

$\int \frac{1}{1+y^2}\,dy=\tan^{-1}(y)+c$

$\int \frac{1}{1+x^2}\,dx=\tan^{-1}x+c$

$\tan^{-1}: (-\infty,+\infty)\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$(f^{-1})'(f(x))=\frac{1}{f'(x)}$

$(\tan^{-1})'(\tan x)=\frac{1}{1+\tan^2 x}$

$(\tan^{-1})'(y)=\frac{1}{1+y^2}$

$$\int \frac{du}{2(u^2+9)} =$$

$$\frac{1}{2} \times \frac{1}{3} \tan\left(\frac{u}{3}\right)$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{x^2}{3}\right) + C$$

$$\int \frac{x}{x^4+9} dx \qquad \text{ـحل}$$

$$u = x^2 \Rightarrow du = 2x\,dx$$

$$\int \frac{1}{x^2+a^2} dx = \int \frac{\frac{1}{a^2}}{\frac{x^2}{a^2}+1} dx$$

$$u = \frac{x}{a}$$

$$du = \frac{1}{a} dx$$

$$= \int \frac{\frac{1}{a} du}{u^2+1}$$

$$= \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x^2+a^2} dx \qquad \text{ـحل}$$

$$\int \frac{1}{x^2+2} dx = ?$$