قضیه اساسی حساب

① $g(x) = \int_a^x f(t)\,dt \Rightarrow$

$g'(x) = f(x)$

② $\int_a^b f(x)\,dx = f(b) - f(a)$

حل:

$$\lim_{x \to 3} \frac{x \int_3^x \frac{\sin t}{t}\,dt}{x-3}$$

$$\lim_{x \to 3} \frac{f(x) - f(3)}{x-3} = f'(3)$$

$$g(x) = \int_1^x \frac{\sin t}{t}\,dt$$

$$g(x) - g(3) = \int_1^x \frac{\sin t}{t}\,dt - \int_1^3 \frac{\sin t}{t}\,dt = \int_3^x \frac{\sin t}{t}\,dt$$

بنابراین حد مورد نظر را می‌توان به صورت زیر نوشت:

$$\lim_{x \to 3} x \frac{g(x) - g(3)}{x-3}$$

$$= \lim_{x \to 3} x \times g'(3) = 3 \times \frac{\sin 3}{3} = \sin 3$$

مثال

فرض کنید

$$f(x)=\int_0^{\sin x}\sqrt{1+t^2}\,dt,$$

$$g(y)=\int_3^{y} f(x)\,dx$$

مطلوب است

$$g''\left(\frac{\pi}{6}\right)=?$$

$$g'(y)=f(y)$$

$$g''(y)=f'(y)$$

مطلوب ما می‌شود

$$f'\left(\frac{\pi}{6}\right)$$

$$f'(x)=\cos x\times\sqrt{1+\sin^2 x}$$

$$f_1=\int_0^{x}\sqrt{1+t^2}$$

$$f_2=\sin(x)$$

$$f(x)=f_1 f_2(x)\Rightarrow f'(x)=f_2'(x)\,f_1'(f_2(x))$$

تذکر

$$\left(f_1 f_2(x)\right)'=f_2'(x)\,f_1'(f_2(x))$$

$$f_1(x)=\int_0^{x}\sqrt{1+t^2}\,dt\Rightarrow f_1'(x)=\sqrt{1+x^2}$$

$$f_2(x)=\sin x\Rightarrow f_2'(x)=\cos x$$

$$\left(\int_0^{f(x)} g(t)\,dt\right)'=$$

$$f'(x)\,g(f(x))$$

$$\int_{h(x)}^{f(x)} g(t)\,dt=?$$

$$\int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \int \frac{1}{\cos^2 x}\,dx = \tan x$$

$$\int \sec x \tan x\,dx = \sec x + C$$

ویژگی‌های انتگرال معین

$$\int c f(x)\,dx = c\int f(x)\,dx$$

$$\int k\,dx = kx + c$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b$$

توجه: انتگرال نامعین

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\int \sqrt{x} = \int x^{\frac{1}{2}}$$

$$\int x^n = \frac{x^{n+1}}{n+1} + C$$

$$\int x^{\frac{1}{2}} = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3} \times x^{\frac{3}{2}}$$

$$\text{مساحت مورد نظر} = \int_1^4 \sqrt{x}\,dx$$

$$= \int_1^4 \left(x^{\frac{1}{2}}\right) dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_1^4$$

$$= \frac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{2}{3}\left(\sqrt{4^3} - 1\right)$$

مثال: مساحت زیر منحنی $y=\sqrt{x}$ را از $x=1$ تا $x=4$ حساب کنید.

لطفا

$$\int_{\frac{1}{2}}^{1} \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$$

$$= \int_{\frac{1}{2}}^{1} \left(2 + t^{\frac{1}{2}} - t^{-2}\right) dt =$$

$$2t + \frac{2}{3} t^{\frac{3}{2}} + t^{-1} \Big|_{\frac{1}{2}}^{1} = \dots$$

dic

$$\int_0^{12} (x - 12\sin x)\, dx$$

$$\frac{x^2}{2} + 12\cos x \Big|_0^{12}$$

$$= \frac{12^2}{2} + 12\cos(12) - 12$$

din

$$\int_0^3 (x^3 - 6x)\, dx$$

$$= \frac{x^4}{4} - \frac{6x^2}{2} \Big|_0^3$$

$$= \frac{3^4}{4} - \frac{6 \times 3^2}{2}$$

din

$$\int \frac{\cos\theta}{\sin^2\theta} = \int \cos\theta \times \csc\theta = -\csc\theta$$

$$\frac{1}{\sin}$$

$$f(\theta) = \frac{1}{\sin\theta}$$

$$f'(\theta) = \frac{-\cos\theta}{\sin^2\theta}$$

$$\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \frac{-1}{\sin\theta}$$

$$\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = ?$$

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$$

روش جایگذاری (تغییر متغیر)

$$\int f'(u)\,du = f(u)$$

$$\int f'(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = f(\underbrace{g(x)}_{u})$$

$u = g(x)$

$du = g'(x)\,dx$

مثال

$$\int 2x\sqrt{1+x^2}\,dx$$

$u = 1+x^2$

$du = 2x\,dx$

$$\int u^{\frac{1}{2}}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + C$$

$$\int 2x\sqrt{1+x^2}\,dx =$$

$$\frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$$

مثال

$$\int x^3\cos(x^4+2)\,dx$$

$u = x^4+2$

$du = 4x^3\,dx$

$$= \frac{1}{4}\sin(x^4+2)$$

$$\int \frac{\cos(u)}{4}\,du$$

$$= \frac{1}{4}\int \cos(u)\,du$$

$$= \frac{1}{4}\sin(u)$$

$$\int \frac{x\,dx}{\sqrt{1-4x^2}} = \frac{-1}{8}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du = -\frac{1\times 2}{8}u^{\frac{1}{2}}+c =$$

$$-\frac{1}{4}\left(1-4x^2\right)^{\frac{1}{2}} = -\frac{1}{4}\sqrt{1-4x^2}+c$$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx$$

$u = 1-4x^2$

$du = -8x\,dx$

$\Rightarrow x\,dx = \frac{du}{-8}$

dir

$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{u}\,du$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{2}\times\frac{2}{3}u^{\frac{3}{2}}$$

$$= \frac{1}{3}u^{\frac{3}{2}} = \frac{1}{3}\sqrt{(2x+1)^3}$$

$$\int \sqrt{2x+1}\,dx$$

$u = 2x+1$

$du = 2\,dx$

dir

$$\int \cos(ax)\,dx =$$

$$\int \frac{\cos(u)\,du}{a} =$$

$$\frac{1}{a}\sin(u) = \frac{1}{a}\sin(ax)$$

$$\int \cos(ax)\,dx$$

$u = ax$

$du = a\,dx$

$$\int \left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right) dx = 2\int \sin u \times du$$

$$= -2\cos(u) = 2\cos\left(\sqrt{x}\right) + C$$

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$$

$\sqrt{x} = u$

$du = \frac{1}{2\sqrt{x}}\, dx$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\left(u^2 + 1 - 2u\right) du$$

$$= \frac{1}{2}\left(\int u^{\frac{5}{2}} + u^{\frac{1}{2}} - 2u^{\frac{3}{2}}\right) du$$

$= \ldots$

$x^4 \times x = (x^2)^2 \times x$

$$\int \sqrt{1+x^2}\; x^5\, dx =$$

$$\int \sqrt{1+x^2}\; (x^2)^2\, x\, dx =$$

$$= \frac{1}{2}\int \sqrt{u}\,(u-1)^2\, du$$

$$\int \sqrt{1+x^2}\; x^5\, dx$$

$u = 1 + x^2$

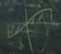
$du = 2x\, dx$

دقت کنید!

$$\int_a^b f'(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du$$

$u = g(x)$

$du = g'(x)\,dx$



روش اول

$$\int_0^4 \sqrt{2x+1}\,dx$$

$u = 2x+1$

$$= \frac{1}{2}\int_1^9 u^{\frac{1}{2}}\,du = \frac{1}{3}u^{\frac{3}{2}}\Big|_1^9$$

روش دوم

$$\int_0^4 \sqrt{2x+1}\,dx$$

$$\frac{1}{3}(2x+1)^{\frac{3}{2}}\Big|_0^4$$

$$\int \sqrt{2x+1}\,dx = \int \frac{u^{\frac{1}{2}}\,du}{2} = \frac{1}{3}u^{\frac{3}{2}} = \frac{1}{3}(2x+1)^{\frac{3}{2}}$$

$$\int_0^4 \sqrt{2x+1}\,dx$$

$u = 2x+1$

$du = 2\,dx$

توجه

فرض کنید f یک تابع فرد باشد

$\forall x \in D_f \quad f(-x) = -f(x)$

در این صورت

$$\int_{-a}^{a} f(t)\,dt = 0$$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$-\int_{0}^{-a} f(x)\,dx = +\int_{0}^{a} f(-u)\,du$$

$u = -x$

$du = -1\,dx$

$$= \int_{0}^{a} -f(u)\,du = -\int_{0}^{a} f(u)\,du$$

$$= -\int_{0}^{a} f(x)\,dx$$

توجه

فرض کنید f یک تابع زوج باشد

$(\forall x \in D_f \quad f(-x) = f(x))$

در این صورت

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

مث

$$\int_0^1 \frac{dx}{(1+\sqrt{x})^4} = \int_1^2 \frac{2(u-1)\,du}{u^4}$$

$1+\sqrt{x} = u$

$du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow dx = 2\sqrt{x}\,du$

$dx = 2(u-1)\,du$

مث

$$\int x\sqrt{1-x^2}\,dx$$

$u = 1-x^2$

$du = -2x\,dx$

$$= \int -\frac{1}{2}u^{\frac{1}{2}}\,du = \ldots$$

$$\int \sin t\sqrt{1+\cos t}\,dt =$$

$$\int -u^{\frac{1}{2}}\,du = -\frac{2}{3}u^{\frac{3}{2}} = -\frac{2}{3}(1+\cos t)^{\frac{3}{2}} + C$$

مث

$$\int \sin t\sqrt{1+\cos t}\,dt$$

$u = 1+\cos t$

$du = -\sin t\,dt$

مث

$$\int_{-2}^{2}(x^6+1)\,dx = 2\int_0^2(x^6+1)\,dx$$

مث

$$\int_{-1}^{1}\frac{\tan x}{1+x^2+x^4}\,dx = 0$$

f یک تابع فرد است:

$$f(x) = \frac{\tan x}{1+x^2+x^4}$$

$$f(-x) = \frac{-\tan x}{1+x^2+x^4} = -f(x)$$