ادامه روش‌ها انتگرال‌گیری

یادآوری

$\tan'(x) = 1 + \tan^2(x) = \sec^2 x$

$(\tan^2 = \sec^2 - 1)$

$\sec'(x) = \left(\frac{1}{\cos x}\right)' = \frac{\sin x}{\cos^2 x} = \boxed{\sec x \tan x}$

مثال

زوج است

$\int \tan^{6} x \sec^{4} x \, dx =$

$\int (\tan^2)^3 ((\sec^2))^2 \, dx =$

$\int (\tan^2)^3 (\sec^2) (1+\tan^2) \, dx$

$u = \tan x$

$\int u^6 (1+u^2) \, du = \cdots$

مثال

فرد

$\int (\tan x)^{5} (\sec x)^{7} \, dx$

$\int (\tan x \sec x) (\tan^2 x)^2 (\sec^2)^3 \, dx$

$= \int (\tan x \sec x) (\sec^2 x - 1)^2 (\sec^2)^3 \, dx$

$u = \sec x$

$\int (u^2 - 1)^2 u^6 \, du = \cdots$

ادامه روشهای انتگرال گیری

① $\int \sec x\,dx = \ln(\sec x + \tan x)$

$\int \sec x\,dx = \int \frac{1}{\cos x}dx = \int \frac{\cos x}{\cos^2 x}dx = \int \frac{\cos x}{1-\sin^2 x}dx$

$= \int \frac{du}{1-u^2}$

مثال

② $\int \sec^2 x\,dx = \tan x$

③ $\int \sec^3 x\,dx = \int \frac{1}{\cos^3 x}dx$

$= \int \sec^2 x \sec x\,dx =$ (u = $\sec x$, dv = $\sec^2 x\,dx$)

$v = \tan x$

$uv - \int v\,du =$

$\sec x \tan x - \int \tan x \sec x \tan x\,dx$

$\int \tan^2 x \sec x\,dx = \int (\sec^2 x - 1)\sec x\,dx$

$= \int \sec^3 x\,dx - \int \sec x\,dx$

$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$\Rightarrow 2\int \sec^3 x\,dx = \sec x \tan x + \ln(\sec x + \tan x)$

فرض کنیم که تابع زیر انتگرال شامل عبارت

$\sqrt{a^2-x^2}$ باشد.

$-a \leq x \leq a$

تغییر متغیر پیشنهادی:

$x = a\sin t$

$\sqrt{x^2-a^2}$

$\sqrt{x^2+a^2}$

$\sqrt{a^2-x^2}$

(این جمله)

می‌رسیم انتگرالها شامل

$$\int \frac{1}{2}\left(\sin(-x) + \sin(9x)\right) dx$$

$= \dots$

مثال

$$\int \sin 4x \cos(5x)\, dx$$

$$\sin a \cos b = \frac{1}{2}\left(\sin(a-b) + \sin(a+b)\right)$$

$$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$

$$\cos a \cos b = \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right)$$

$$\int \sqrt{9-x^2}\, dx = \int \sqrt{9-9\sin^2 t} \times 3\cos t\, dt$$

$9(1-\sin^2) = 9(\cos^2)$

$$= \int 3\cos t \times 3\cos t\, dt = 9\int \cos^2 t\, dt =$$

$$9\int \frac{1+\cos(2t)}{2} dt = 9\left(\frac{1}{2}t + \frac{1}{4}(\sin 2t)\right)$$

$$t = \sin^{-1}\left(\frac{x}{2}\right) + C$$

$$\int \sqrt{9-x^2}\, dx$$

$9-x^2 \geq 0 \quad -3 \leq x \leq 3$

$x = 3\sin t, \; dx = 3\cos t\, dt$

$$\int \frac{1}{\sqrt{4-4\sin^2 t}} \times 2\cos t\, dt =$$

$4(1-\sin^2) = 4\cos^2 t$

$$= \int \frac{2\cos t\, dt}{2\cos t} = \int dt =$$

$dx = a\cos t\, dt$

$$\int \frac{1}{\sqrt{4-x^2}}\, dx$$

$x = 2\sin t \Rightarrow t = \sin^{-1}\left(\frac{x}{2}\right)$

$dx = 2\cos t\, dt$

$$\int \frac{(2\sin t-1)\,2\cos t\,dt}{2\cos t} =$$

$$\int 2\sin t\,dt - \int dt = -2\cos(t) - t + c$$

$$= -2\cos\left(\sin^{-1}\left(\tfrac{u}{2}\right)\right) - \sin^{-1}\left(\tfrac{u}{2}\right) + C =$$

$$-2\cos\left(\sin^{-1}\left(\tfrac{x+1}{2}\right)\right) - \sin^{-1}\left(\tfrac{x+1}{2}\right) + C$$

$$\int \frac{(u-1)\,du}{\sqrt{4-u^2}} \qquad \frac{u}{2} = \sin t \Rightarrow t = \sin^{-1}\left(\frac{u}{2}\right).$$

$$u = 2\sin t \qquad du = 2\cos t\,dt$$

$$\sqrt{4-u^2} = \sqrt{4-4\sin^2 t} = 2\cos t$$

$$\int \frac{x\,dx}{\sqrt{4-(x+1)^2}}$$

$$u = x+1$$

$$du = dx$$

$$9\left(\frac{1}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{1}{4}\sin\left(2\sin^{-1}\left(\frac{x}{3}\right)\right)\right) + C$$

$$\int \frac{x\,dx}{\sqrt{3-2x-x^2}} =$$

$\underline{dw}$

$$\int \cos t \left(\frac{1}{\cos^2 t}\right) dt = \int \sec t \, dt = \ln\left(\sec t + \tan t\right) = \ln\left(\sec\left(\tan^{-1}\left(\frac{x}{2}\right)\right) + \tan\left(\tan^{-1}\frac{x}{2}\right)\right) + C$$

$$\frac{x}{2}$$

$$\int \frac{1}{\sqrt{x^2+4}} \, dx = \int \frac{1}{\sqrt{4\tan^2 t + 4}} \times 2(1+\tan^2 t) \, dt$$

$$= \int \frac{1}{\frac{2}{\cos t}} \times 2\left(1+\tan^2 t\right) dt =$$

مثال

$$\int \frac{1}{\sqrt{x^2+4}} \, dx$$

$$x = 2\tan t$$

$$dx = 2\left(1+\tan^2 t\right)$$

② أشكالها شكل

$$\sqrt{x^2+a^2}$$

$$x = a\tan t$$

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2 t + a^2} = a\sqrt{1+\tan^2 t} = \frac{a}{\cos t}$$

$$dx = a\left(1+\tan^2 t\right)$$

$$\frac{\left(\frac{3}{2}\right)^3 \times 3}{3^3 \times 2} \int \tan^3\theta \cos\theta \, d\theta$$

$$= \int \frac{\left(\frac{3}{2}\tan\theta\right)^3}{\left(\frac{3}{\cos\theta}\right)^3} \times \frac{3\left(1+\tan^2\theta\right) d\theta}{2} \quad \left(\frac{1}{\cos^2}\right)$$

$$= \int \frac{\left(\frac{3}{2}\right)^3 \tan^3\theta \times 3}{\left(\frac{1}{\cos\theta}\right)^3 \times 3^3 \times 2} \, d\theta =$$

$$= \int \frac{x^3 \, dx}{\left(\sqrt{(2x)^2+9}\right)^3}$$

$$2x = 3\tan\theta$$

$$2dx = 3\left(1+\tan^2\theta\right) d\theta$$

تمرین

$$\int \sqrt{x^2+9} \, dx$$

مثال

$$\int \frac{x^3 \, dx}{\left(4x^2+9\right)^{\frac{3}{2}}} = \int \frac{x^3 \, dx}{\left(\sqrt{4x^2+9}\right)^3}$$

تمرین

انتگرال بالا را با تغییر متغیر

$$x = 2\sinh t$$

حل کنید.

$$\int \frac{x^2 x\,dx}{\left(\sqrt{4x^2+9}\right)^3} = \int \frac{\left(\frac{u-9}{4}\right)\frac{du}{8}}{u^{\frac{3}{2}}}$$

$= \dots$

$$\int \frac{x^3\,dx}{\left(\sqrt{4x^2+9}\right)^3}$$

$u = 4x^2+9$ $du = 8x\,dx$

$x^2 = \frac{u-9}{4}$

$u = \cos\theta$

$$\int \sin\theta \left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) d\theta$$

$$= -\int \left(\frac{1-u^2}{u^2}\right) du = \dots$$

$$\int \tan^3\theta \cos\theta \, d\theta = \int \frac{\sin^3\theta}{\cos^2\theta} d\theta$$

$$= \int \frac{\sin\theta(\sin^2\theta)}{\cos^2\theta} d\theta =$$

حل:

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

$$x = 2\tan\theta$$

$$dx = 2(1+\tan^2\theta) = \frac{2}{\cos^2\theta}$$

$$\int \frac{1}{4\tan^2\theta \;\frac{2}{\cos\theta}} \times \frac{2}{\cos^2\theta}\,d\theta$$

$$= \int \frac{1}{4\tan^2\theta} \times \frac{1}{\cos\theta}\,d\theta$$

$$\frac{1}{4}\int \frac{\cos^2\theta}{\sin^2\theta} \times \frac{1}{\cos\theta}\,d\theta =$$

$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \dots$$

③ حالت: $\sqrt{x^2-a^2}$

$x \leq -a$ یا $x \geq a$

$$x = a\sec\theta$$

$$x = a\cosh\theta$$

$$\int \frac{\sec^2\theta - 1}{\sec\theta}\, d\theta =$$

$$\int \sec\theta\, d\theta - \int \cos\theta\, d\theta$$

$$= \cdots$$

$$\int \frac{\sqrt{3}\tan\theta}{3\sec^2\theta}\left(\sqrt{3}\sec\theta\tan\theta\right) d\theta$$

$$= \int \frac{\tan^2}{\sec\theta}\, d\theta =$$

$$\sqrt{x^2-3} = \sqrt{3\sec^2\theta - 3} =$$

$$\sqrt{3}\sqrt{\sec^2\theta - 1} = \sqrt{3}\tan\theta$$

$$(1+\tan^2 = \sec^2)$$

$$\int \frac{\sqrt{x^2-3}}{x^2}\, dx$$

$$x = \sqrt{3}\sec\theta$$

$$dx = \sqrt{3}\sec\theta\tan\theta$$

di

$$\sqrt{a^2-x^2} \Rightarrow x = a\sin\theta$$

$$\sqrt{a^2+x^2} \Rightarrow x = a\tan\theta \;\text{ or }\; x = a\sinh t$$

$$\sqrt{x^2-a^2} \Rightarrow x = a\sec\theta \;\text{ or }\; x = a\cosh t$$

$$(\tanh)' = 1 - \tanh^2$$

$$\Rightarrow = -\int -\tanh^2 dt = -\int \left((1 - \tanh^2 t) - 1 \right) dt$$

$$= -\tanh t + t + C = -\tanh\left(\cosh^{-1}\left(\frac{x}{\sqrt{3}}\right)\right) + \cosh^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$\int \frac{\sqrt{x^2-3}}{x^2}\, dx = \int \frac{\sqrt{3\cosh^2 t - 3}}{3\cosh^2 t} \times \sqrt{3}\sinh t\, dt$$

$$= \int \frac{\sinh t \ \sinh t\, dt}{\cosh^2 t} = \int \tanh^2 t\, dt$$

$$\int \frac{\sqrt{x^2-3}}{x^2}\, dx$$

$$x = \sqrt{3}\cosh t \Rightarrow t = \cosh^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

$$dx = \sqrt{3}\ \sinh t$$

dix