روش تغییر متغیر

در یک انتگرال نامعین با استفاده از روش تغییر متغیر

$$\int f'(u)\,du = f(u) + c \qquad \boxed{\int f'(x)\,dx = f(x) + c}$$

$$\int f'(g(x))\,g'(x)\,dx = \int f'(u)\,du = f(g(x)) + c$$

$u = g(x)$

$du = g'(x)\,dx$

مثال

$$\int \frac{\sin(\ln x)}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int \frac{\sin(\ln x)}{x}\,dx = \int \sin(u)\,du =$$

$$-\cos(u) + c \xrightarrow{\text{جایگذاری}} -\cos(\ln x) + c$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

$u = \cos x$

$du = -\sin x\,dx$

$$\int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u} = -\ln|u| \xrightarrow{\text{جایگذاری}} -\ln|\cos x|.$$

$$\textcircled{2}\int \frac{1}{1-x^2}\,dx = \int \left(\frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)\right)dx$$

$$= \frac{1}{2}\int \frac{1}{1+x}\,dx + \frac{1}{2}\int \frac{1}{1-x}\,dx$$

$\cosh^{-1}x$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\textcircled{1}\int \frac{1}{1-x^2}\,dx = \tanh^{-1}(x) + C$$ eg

$u = \sin x \Rightarrow du = \cos x\,dx$

$$\int \frac{\cos x}{1-\sin^2 x}\,dx = \int \frac{du}{1-u^2} = \tanh^{-1}(u) + C$$

$$= \tanh^{-1}(\sin(x)) + C = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + C$$

$$\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \quad \text{de}$$

$$\int \frac{\cos x}{\cos x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx$$

$$\int \sec^2 x\,dx = \int \frac{1}{\cos^2 x}\,dx = \tan x + C$$

$$\tanh^{-1}(t) = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$$

① $\int \frac{1}{1-x^2}\,dx = \tanh^{-1}(x) + C$

② $\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$

$$\int \frac{1}{x-1}\,dx = \ln|x-1| + C$$

$$\Rightarrow \int \frac{1}{1-x^2}\,dx = \frac{1}{2}\left[\ln|x+1| - \ln|x-1| + C\right]$$

$$= \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$$

$$\int \frac{1}{1+x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C =$$

$u = 1+x$

$du = dx$

$$\ln|x+1| + C$$

$$\int \frac{\sin x}{\cos^2 x}\,dx = \sec x + C = \frac{1}{\cos x} + C$$

تمرین

$$\int \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= -\int u^{-2}\,du = \frac{-u^{-1}}{-1} + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$= \frac{1}{u} + C = \frac{1}{\cos x} + C$$

مثال

$$\int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{-du}{u^2}$$

$u = \cos x$

$du = -\sin x\,dx$

تمرین

$$\int \sec^3 x\,dx = \int \frac{1}{\cos^3 x}\,dx = ?$$

(در جلسات بعدی درباره‌ی این صحبت می‌کنیم)

$= \tan^{-1}(x) + \frac{1}{2}\ln|1+x^2| + C$

$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|1+x^2| + C$

$\int \frac{1+x}{1+x^2}\,dx$ $\quad du$

$= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{(1+x^2)}\,dx$

u

$du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

$\int x e^{-x^2}\,dx = \int \frac{(e^u)\,du}{-2}$ $\quad du$

$u = -x^2$

$du = -2x\,dx$

$= -\frac{1}{2}\int e^u\,du =$

$-\frac{1}{2}\left(e^u + c\right) = -\frac{1}{2}\left(e^{-x^2} + c\right)$

$u = \cos x$

$du = -\sin x\,dx$

$\int \frac{\sin x}{1+\cos^2 x}\,dx = \int \frac{-du}{1+u^2} = -\tan^{-1}(u) + C =$

$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$ $\quad -\tan^{-1}(\cos x) + C$

$$\frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

$$\int \frac{1}{u^2-1}\,du =$$

$$\frac{1}{2}\int\left(\frac{1}{u-1} - \frac{1}{u+1}\right)du =$$

$$\int \frac{1}{x^2+2x}\,dx = \int \frac{1}{(x+1)^2-1}\,dx$$

$$u = x+1$$

$$du = dx$$

تمرين

$$\int \frac{(\arctan x)^2}{x^2+1}\,dx = ?$$

$$\int (1-2x)^9\,dx = ?$$

$$\int \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv} = uv - \int v\,du$$

مثال

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$v = x \qquad dv = dx$

$$= x\ln x - \int x\cdot\frac{1}{x}\,dx = \boxed{x\ln x - x + C}$$

روش انتگرال‌گیری جزء به جزء

(Integration by Parts)

$$d(uv) = u\,dv + v\,du$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$$

تمرین

$$\int (\sin t)\sqrt{1+\cos t}\;dt$$

Maple

Matlab

راه حل دوم

$$\int \frac{1}{x^2+2x}\,dx = \frac{1}{2}\int\left(\frac{1}{x} - \frac{1}{x+2}\right)dx = \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2|$$

$$\frac{1}{x} - \frac{1}{x+2} = \frac{x+2-x}{x^2+2x} = \frac{2}{x^2+2x}$$

$$\int \underbrace{x}_{u}\underbrace{\sin x\,dx}_{dv} = x(-\cos x) - \int(-\cos x)\,dx =$$

$$-x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

$$\int \underbrace{x^2}_{u}\underbrace{\sin x\,dx}_{dv} =$$

$$\sin x \cdot \frac{x^2}{2} - \int \frac{x^2}{2}\cos x\,dx$$

$$(u(x)v(x))' = u'(x)v(x) + v'(x)u(x)$$

$$\left(u(x)v(x)\right)'dx = u'(x)v(x)dx + v'(x)u(x)dx$$

$$d(uv) = v\,du + u\,dv$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{\ln x}_{u}\underbrace{dx}_{dv} = \ln x \cdot x - \int x \cdot \frac{1}{x}\,dx = \cdots$$

$$v = x \Rightarrow dv = 1\,dx$$