where φ is in mth-order logic. The subscript n stands for the number of blocks of quantifiers. $\mathbf{Q}$ is either $\exists$ or $\forall$ depending on whether n is odd or even. If the formula has n blocks of $(m+1)$th-order quantifiers beginning with a universal quantifier, it is called a $\mathbf{\Pi}_n^m$ formula. As we have explained in our introduction to this book, we shall almost always deal only with first-order logic. The $\mathbf{\Sigma}_n^0$ and $\mathbf{\Pi}_n^0$ formulas are first-order, and we have already introduced them in Section 3.1.

Exercises

4.1.1. Let D be a filter over I. Show that the following are equivalent:
(i). D is a proper filter.
(ii). $0 \notin D$.
(iii). D has the finite intersection property.

4.1.2.
(i). The intersection of any set of filters over I is a filter over I.
(ii). The union of any chain of proper filters over I is a proper filter over I.

4.1.3. Let D be an ultrafilter over I and let $X \in D$. Then $D \cap S(X)$ is an ultrafilter over X. Similarly for proper filters.

4.1.4. D is a principal ultrafilter over I if and only if $D = \{X \in S(I) : i \in X\}$ for some $i \in I$.

4.1.5. If X is infinite, then there exist nonprincipal ultrafilters over X.

4.1.6. A filter D is principal if and only if $\bigcap D \in D$. Every filter over a finite set is principal.

4.1.7. Let D be a proper filter over I. D is an ultrafilter if and only if for all X, $Y \in S(I)$, $X \cup Y \in D$ implies $X \in D$ or $Y \in D$.

4.1.8. Let E be a countable subset of $S(\omega)$. Then the filter generated by E cannot be a nonprincipal ultrafilter.

4.1.9. Prove Propositions 4.1.5 and 4.1.7.

4.1.10. Suppose D is the principal ultrafilter where $\{j\} \in D$. Prove that $\prod_D \mathfrak{A}_i$ is isomorphic to $\mathfrak{A}_j$.

4.1.11. Let D be a proper filter over I, let $X \in D$, and let $E = D \cap S(X)$.

Prove that

$$\prod_D \mathfrak{A}_i \cong \prod_E \mathfrak{A}_x.$$

(Cf. Exercise 4.1.3.)

4.1.12. The *direct product* of the models $\mathfrak{A}_i$, $i \in I$ is defined as follows. The universe set is the Cartesian product $\prod_{i \in I} A_i$. A relation $S(f^1 \dots f^n)$ holds in the direct product if and only if the corresponding relation $R_i(f^1(i) \dots f^n(i))$ holds in $\mathfrak{A}_i$ for all $i \in I$. The functions H are defined by

$$H(f^1 \dots f^n) = \langle G_i(f^1(i) \dots f^n(\mathrm{i})) : i \in I \rangle,$$

and the constants b are defined by

$$b = \langle a_i : i \in I \rangle.$$

Prove that the direct product is isomorphic to the trivial reduced product $\prod_{\{I\}} \mathfrak{A}_i$.

4.1.13. If D, E are proper filters over I and $D \subset E$, then $\prod_E \mathfrak{A}_i$ is a homomorphic image of $\prod_D \mathfrak{A}_i$. Hence every reduced product is a homomorphic image of the direct product of the models $\mathfrak{A}_i$. (See Section 2.1 for the definitions of homomorphism and homomorphic image.)

4.1.14. Show that there exists an ultraproduct $\prod_D A_i$ of finite sets A_i which is infinite.

4.1.15. Let D be a proper filter over I. If each $\mathfrak{A}_i$ is isomorphically embedded in $\mathfrak{B}_i$, then $\prod_D \mathfrak{A}_i$ is isomorphically embedded in $\prod_D \mathfrak{B}_i$. If each $\mathfrak{A}_i$ is isomorphic to $\mathfrak{B}_i$, then $\prod_D \mathfrak{A}_i$ is isomorphic to $\prod_D \mathfrak{B}_i$. If each $\mathfrak{A}_i$ is a homomorphic image of $\mathfrak{B}_i$, then $\prod_D \mathfrak{A}_i$ is a homomorphic image of $\prod_D \mathfrak{B}_i$.

4.1.16. Let D be an ultrafilter over I. If $\mathfrak{A}_i \equiv \mathfrak{B}_i$ for all $i \in I$, then $\prod_D \mathfrak{A}_i \equiv \prod_D \mathfrak{B}_i$. If $\mathfrak{A}_i$ is elementarily embedded in $\mathfrak{B}_i$ for all $i \in I$, then $\prod_D \mathfrak{A}_i$ is elementarily embedded in $\prod_D \mathfrak{B}_i$.

4.1.17. A class K of models for $\mathscr{L}$ is said to be a *pseudo-elementary class* iff for some expansion $\mathscr{L}'$ of $\mathscr{L}$ and some elementary class K' for $\mathscr{L}'$, K is the class of all reducts of models in K' to $\mathscr{L}$. Prove that every pseudo-elementary class is closed under ultraproducts. Prove that if Γ is a set of $\mathbf{\Sigma}_1^1$ sentences, then the class of all models of Γ is a pseudo-elementary class.

4.1.18. Let K be a class of models for $\mathscr{L}$. Let M be the class of all models $\mathfrak{A}$ such that $\mathfrak{A}$ is elementarily equivalent to an ultraproduct of members of K. Prove that M is an elementary class, and is the least elementary class which includes K.

4.1.19. Give the proof of Theorem 4.1.12 (ii).

4.1.20. The following are equivalent:

(i). K is a basic elementary class.

(ii). There exists a finitely axiomatizable theory T in $\mathscr{L}$ such that K is the class of all models of T.

(iii). Both K and its complement are elementary classes.

4.1.21. Let D be a principal ultrafilter. Prove that $d(\mathfrak{A}) = \prod_D \mathfrak{A}$, whence d is an isomorphism of $\mathfrak{A}$ onto $\prod_D \mathfrak{A}$.

4.1.22. Let D be a proper filter. Prove that d is an isomorphic embedding of $\mathfrak{A}$ into $\prod_D \mathfrak{A}$.

4.1.23. Let K and M be two classes of models. Let T_1 be the theory of K (the set of all sentences which hold in every model in K), and let T_2 be the theory of M. Prove that $T_1 \cup T_2$ is consistent if and only if some ultraproduct of members of K is elementarily equivalent to some ultraproduct of members of M.

4.1.24. Let $\mathfrak{A}_\alpha$, $\alpha < \beta$, be an elementary chain of length $\beta > 0$. Let D be an ultrafilter over β such that for each $\alpha < \beta$, the set $\{\gamma : \alpha \leqslant \gamma < \beta\}$ belongs to D. Prove that $\bigcup_{\alpha<\beta} \mathfrak{A}_\alpha$ is elementarily embedded in the ultraproduct $\prod_D \mathfrak{A}_\alpha$.

4.1.25. In the above exercise suppose that $\mathfrak{A}_\alpha$, $\alpha < \beta$, is only a chain of models, and D is only a proper filter. Prove that $\bigcup_{\alpha<\beta} \mathfrak{A}_\alpha$ is isomorphically embedded in the reduced product $\prod_D \mathfrak{A}_\alpha$.

4.1.26. Using Theorem 4.1.12, prove that none of the following theories is finitely axiomatizable:

(i) infinite models of pure identity theory;

(ii) fields of characteristic zero;

(iii) real closed fields;

(iv) algebraically closed fields;

(v) divisible Abelian groups;

(vi) torsion-free Abelian groups;

(vii) the theory of the model $\langle \omega, S \rangle$, where S is the successor function.

4.1.27. Show that none of the following classes of models is closed under elementary equivalence (use Corollary 4.1.10):

(i) the class of free groups;
(ii) the class of torsion groups;
(iii) the class of simple groups;
(iv) the class of all rings isomorphic to polynomial rings over the field of rational numbers.

4.1.28. Prove that every model $\mathfrak{A}$ can be isomorphically embedded in some ultraproduct of finite submodels of $\mathfrak{A}$. (For this problem, assume that $\mathscr{L}$ has no function or constant symbols, so we can be sure that finite submodels exist.) This gives a stronger form of Corollary 2.1.9.

4.1.29. Give an example of a $\mathbf{\Pi}_1^1$ formula which is not preserved under ultraproducts.

4.1.30*. Let F_i, $i \in I$ be a family of fields. From the direct product $R = \prod_{i \in I} F_i$. Thus R is a ring. For each ultrafilter D over I, let

$$M_D = \{f \in R : \{i \in I : f(i) = 0\} \in D\}.$$

Prove the following:

(i). For each D, M_D is a maximal ideal in R.

(ii). For every maximal ideal M in R, there is an ultrafilter D over I such that $M = M_D$.

(iii). The ultraproduct $\prod_D F_i$ is isomorphic to the quotient field R/M.

Thus ultraproducts of fields are essentially the same thing as quotient fields of direct products of fields. Show that the same results hold for division rings (which have all the field axioms except for commutativity of multiplication).

4.1.31. Let D be an ultrafilter and let $\mathfrak{A} \times \mathfrak{B}$ be the direct product of $\mathfrak{A}$ and $\mathfrak{B}$. Prove that $\prod_D (\mathfrak{A} \times \mathfrak{B}) \cong \prod_D \mathfrak{A} \times \prod_D \mathfrak{B}$.

4.1.32. Let D be a nonprincipal ultrafilter over ω. Prove that for every infinite set A, the natural embedding $D : A \to \prod_D A$ is a proper embedding. Prove the same result under the weaker hypothesis that the index set I of D can be partitioned into countably many sets none of which belongs to D. (Such ultrafilters are called *countably incomplete* in Section 4.3.)

4.1.33. Let D be a nonprincipal ultrafilter over ω. Prove that the ultrapower $\prod_D \langle \omega, \leqslant \rangle$ is isomorphic to a proper initial segment of $\prod_D (\prod_D \langle \omega, \leqslant \rangle)$. [*Hint*: Use the fact that $\langle \omega, \leqslant \rangle$ is isomorphic to an initial segment of $\prod_D \langle \omega, \leqslant \rangle$.]

4.1.34*. Let $\mathfrak{A}$ be a model with the property that each subset U of A is a relation of $\mathfrak{A}$ and each function $f: A \to A$ is a function of $\mathfrak{A}$. Suppose $\mathfrak{A} \prec \mathfrak{B}$ and there is an element $b \in B$ such that $\mathfrak{B}$ has no proper submodels containing b. Prove that there is an ultrafilter D over A such that $\mathfrak{B} \cong \prod_D A$.

4.1.35*. Let D be an ultrafilter over a set I and let $f: I \to I$. Prove that there is a set $X \in D$ such that either $f(i) = i$ for all $i \in X$, or $f(i) \notin X$ for all $i \in X$. [*Hint*: Let $J = \{i \in I : f(i) \neq i\}$. Show that J can be partitioned into 3 sets X_1, X_2, X_3 such that for each $n \in \{1, 2, 3\}$, $f(i) \notin X_n$ for all $i \in X_n$.]

4.2. Measurable cardinals

In this section we shall study ultraproducts of a special kind. These are ultraproducts where the ultrafilter is α-complete. We shall apply these ultraproducts to the problem of the existence of α-complete ultrafilters. This problem has had a profound influence on the recent development of set theory.

Let α be an infinite cardinal. A filter D over I is said to be *α-complete* iff the intersection of any non-empty set of fewer than α elements of D belongs to D, that is,

$$E \subset D \quad \text{and} \quad |E| < \alpha \quad \text{implies} \quad \bigcap E \in D.$$

We see at once that:

PROPOSITION 4.2.1.

(i). *Every filter is ω-complete.*

(ii). *A filter D is α-complete for all α if and only if D is principal.*

(iii). *If $\alpha < \beta$, then every β-complete filter is α-complete.*

A slightly less trivial proposition is the following:

PROPOSITION 4.2.2. *Let D be a filter over a set I of power α. If D is α^+-complete, then D is principal.*